q-de Rham complex

§ de Rham cx:

A ring, S A-alg.

$\rightsquigarrow$ $\Omega^\bullet_{S/A}$ — initial A-dgа $/S$ — $d : S \to \Omega^1_{S/A}$ univ A-lin deriv.

$$\Omega^j_{S/A} = \Lambda^j \Omega^1_{S/A}$$

$$d : \Omega^j_{S/A} \longrightarrow \Omega^{j+1}_{S/A}$$

$$a\, db_1 \wedge \dots \wedge db_j \longmapsto da\, db_1 \wedge \dots \wedge db_j$$

$\rightsquigarrow$ intrinsic definition $\to$ functorial $\to$ sheafify

If S sm$/A$ $\Longrightarrow$ $\exists$ (loc on S) $A[x_1, \dots, x_n] \xrightarrow[\text{ét}]{} S$

in this case

$$\Omega^j_{S/A} = \bigoplus_{1 \le i_1 < \dots < i_j \le n} S\, dx_{i_1} \wedge \dots \wedge dx_{i_j}$$

$\to$ computable

Thm (Grothendieck)

X sm $/ \mathbb{C}$

$$\Rightarrow \quad H^i(X(\mathbb{C})^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i(X, \Omega^{\cdot}_{X/\mathbb{C}})$$

[uses $\mathbb{C} \to \Omega^{\cdot}_{X(\mathbb{C})^{an}/\mathbb{C}}$ res. + GAGA]

In part if $X = \operatorname{Spec} S$

$$\Rightarrow \quad H^i(X(\mathbb{C})^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = \frac{\operatorname{Ker}(\Omega^i_{S/\mathbb{C}} \xrightarrow{d} \Omega^{i+1}_{S/\mathbb{C}})}{\operatorname{Im}(\Omega^{i-1}_{S/\mathbb{C}} \xrightarrow{d} \Omega^i_{S/\mathbb{C}})}$$

Have $H^i(X(\mathbb{C})^{an}, \mathbb{Z}_p) = H^i_{et}(X, \mathbb{Z}_p)$

$\exists$? description via differential forms if X is not over $\mathbb{C}$ but some p-adic ring.

§ Example of q-de Rham

$$\mathbb{Z}_p[q] \ni [n]_q = \frac{q^n - 1}{q-1} = 1 + q + \dots + q^{n-1}$$

consider small nbhd around $q-1$

$$\begin{tikzcd}
\mathbb{Z}_p & \mathbb{Z}_p[q] \arrow[l, "p^r \mapsfrom q"'] \arrow[r] & \mathbb{Z}_p[[q-1]] \arrow[r, "q \mapsto 1"] & \mathbb{Z}_p \\
|\mathbb{P}^{n-1}(\mathbb{F}_{p^r})| & {[n]_q} \arrow[l, mapsto] \arrow[r, mapsto] & {[n]_q} \arrow[r, mapsto] & n
\end{tikzcd}$$

" q-dR cx is a deformation of the dR$/\mathbb{Z}_p$ to $\mathbb{Z}_p[[q-1]]$ "

(goes back to Jackson (1908) / Aomoto 1990)

let $R = \mathbb{Z}_p[x]^\wedge$

<u>q-dR of R:</u>

$$\nabla_q : R[[q-1]] \longrightarrow R[[q-1]]\, dx$$

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{qx - x}\, dx$$

$$[\to f'(x)dx \quad (q \to 1)]$$

<u>e.g.</u> $$\nabla_q(x^n) = \frac{(qx)^n - x^n}{qx - x}\, dx = \left((qx)^{n-1} + (qx)^{n-2}\cdot x + \dots + (qx)x^{n-2} + x^{n-1}\right) dx$$

$$= [n]_q\, x^{n-1}\, dx$$

$$\Rightarrow \quad H^1(\nabla_q) = \mathrm{Coker}\, \nabla_q = \left(\bigoplus_{n \geq 0} \frac{\mathbb{Z}_p[q-1]}{[n]_q} \right)^{\wedge}$$

<u>Main problem</u>: <u>Not functorial</u> !

indeed

$$\begin{array}{ccc} R[\![q-1]\!] & \xrightarrow{\nabla_q} & R[\![q-1]\!]\,dx \\ \downarrow & & \vdots \quad \text{Assum } \exists\ \Theta \ \ \mathbb{Z}_p[\![q-1]\!]\text{-lin} \\ R[\![q-1]\!] & \xrightarrow{\nabla_q} & R[\![q-1]\!]\,dx \end{array} \qquad \begin{array}{c} x \\ \downarrow \\ x+1 \end{array}$$

$$\begin{array}{ccc} x^n & \longmapsto & [n]_q\, x^{n-1}\,dx \\ \downarrow & & \\ (x+1)^n = \sum_{j=0}^{n} \binom{n}{j} x^j & \longmapsto & \sum_{j=1}^{n} \binom{n}{j} [j]_q\, x^{j-1}\,dx \end{array}$$

$$\Rightarrow \quad \Theta(x^{n-1}dx) \text{ should be } \underbrace{\left(\sum_{j=1}^{n} \binom{n}{j} \frac{[j]_q}{[n]_q} x^{j-1} \right) dx}_{\text{not } R[\![q-1]\!]}$$

"An intrinsically defined object such as dR-cx is missing"

§ q-de Rham for a framing

① Standard fact for étale maps

(EGA, IV, Thm 18.1.2, SP, Tag 04DZ)

Assume $\mathrm{Spec}(C_0) \to \mathrm{Spec}(C)$ is a univ homeom.

(e.g. $C \to C_0$ is surj with nilp kernel)

Then

$$(\text{étale } C\text{-alg}) \longrightarrow (\text{étale } C_0\text{-alg})$$

$$R \longmapsto R \otimes_C C_0$$

is an equivalence of cats.

<u>Cor</u>: S p-completely smooth / $\mathbb{Z}_p$, i.e.

$S = \varprojlim_n S/p^n S$, p-tors free and S/pS sm / $\mathbb{F}_p$

Then (after shrinking S Zar loc around a given pt)

$\exists$ <u>framing</u> i.e. $\square : \mathbb{Z}_p[x_1 \dots x_n]^\wedge \longrightarrow S$
p-completely étale
(i.e. $\mathbb{F}_p[x_1 \dots x_n] \to S/pS$ ét)

<u>Pf</u>: $\exists$ $\mathbb{F}_p[x_1 \dots x_n] \longrightarrow S/pS$ ét

use ① above to lift successively to $\mathbb{Z}/p^n[\underline{x}] \longrightarrow S/p^n S$ $\square$

(2) <u>Def-lem:</u> let $[\![\ ]\!] : \mathbb{Z}_p[x_1 \dots x_n]^\wedge \xrightarrow{\text{ét}} S$

$\rightarrow$ get framing

$$\mathbb{Z}_p[q-1, x_1 \dots x_n]^\wedge_{(p,q-1)} \longrightarrow S[\![q-1]\!]$$

let $\gamma_i : \mathbb{Z}_p[q, x_1 \dots x_n] \longrightarrow \mathbb{Z}_p[q, x_1 \dots x_n]$

$$q \longmapsto q$$
$$x_j \longmapsto x_j \quad (j \neq i)$$
$$x_i \longmapsto q x_i$$

Then

$\exists!\ \gamma_i : S[\![q-1]\!] \longrightarrow S[\![q]\!]$

comp with γ_i from above via framing.

<u>Pf.</u> $R = \mathbb{Z}_p[q, x_1 \dots x_n]$, $I = (p, q-1)$

$\rightarrow \gamma_i : \dfrac{R}{I^N} \longrightarrow \dfrac{R}{I^N}$ isom (as $q \in (R/I^N)^\times$

Have

$$\begin{tikzcd}
S[\![q-1]\!]/I \arrow[r, dashed, "\mathrm{id}"] & S[\![q-1]\!]/I \\
R/I \arrow[u] \arrow[r, "\gamma_i = \mathrm{id}"] & R/I \arrow[u]
\end{tikzcd}$$

Assume

$$\begin{tikzcd}
S[\![q-1]\!]/I^{N-1} \arrow[r, dashed, "\exists!\,\gamma_i"] & S[\![q-1]\!]/I^{N-1} \\
R/I^{N-1} \arrow[u, "\text{ét}"] \arrow[ur, "\text{ét}"] \arrow[r, "\gamma_i", "\cong"'] & R/I^{N-1} \arrow[u, "\text{ét}"']
\end{tikzcd}
\overset{(1)}{\Longrightarrow}
\begin{tikzcd}
S[\![q-1]\!]/I^N \arrow[r, dashed, "\exists!"] & S[\![q-1]\!]/I^N \\
R/I^N \arrow[u] \arrow[ur, "\text{ét}"] \arrow[r, "\gamma_i"] & R/I^N \arrow[u, "\text{ét}"']
\end{tikzcd}$$

$\square$

3 Def (framed q-de Rham)

$\square: \mathbb{Z}_p[x_1 \dots x_n]^\wedge \xrightarrow{\text{ét}} S$, $A = \mathbb{Z}_p[\![q-1]\!]$

define $\left(q\text{-}\Omega^{*,\square}_{S[\![q-1]\!]/A}, \nabla_q\right)$ via.

$$q\,\Omega^{j,\square}_{S[\![q-1]\!]/A} := \bigoplus_{1 \le i_1 < \dots < i_j \le n} S[\![q-1]\!]\, dx_{i_1} \wedge \dots \wedge dx_{i_j}$$

and.

$$\nabla_q \left(f\, dx_{i_1} \wedge \dots \wedge dx_{i_j} \right) = \sum_{l=1}^{n} \nabla_{q,l}(f) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_j}$$

$$\text{where } \nabla_{q,l}(f) = \frac{\gamma_l(f) - f}{q x_l - x_l}\, dx_l$$

(with γ_l as in 2)

Depends on choice of framing

Prop: $q\,\Omega^{*,\square}_{S[\![q-1]\!]/A} \otimes^{\mathbb{Z}_p}_{A,\, q-1 \mapsto 0} = \left(\Omega^*_{S/\mathbb{Z}_p}\right)^\wedge$

Conj (Scholze) $q\,\Omega^{*,\square}_{S[\![q-1]\!]/A} \in D_{comp}(A)$

is independent of the choice of $\square$

Georg.

derived $(p, q-1) \overset{\downarrow}{=} (p, [p]_q)$-comp.

(Emil) $\triangleq$ C^* with $C = R\lim_n \left(C \otimes \frac{A}{I^n}\right)$

(p, $q-1$ reg seq in I)

More precisely ↻

(ctd) $\exists$ sym monoidal functor

$$(\text{formally sm } \mathbb{Z}_p\text{-alg}) \longrightarrow D_{comp}(A)$$

$$S \longmapsto q\Omega_S$$

s.t. whenever S admits a framing, then

$$q\Omega_S \cong q\Omega^{*,\square}_{S[q-1]/A}$$

<u>Compare to the following classical picture:</u>

R_0 sm$/\mathbb{F}_p$, $\square : \mathbb{F}_p[x_1, \dots x_n] \xrightarrow{et} R_0$

① $\Rightarrow \exists!$ lift $\mathbb{Z}_p[x_1 \dots x_n]^\wedge \longrightarrow R$

Then $\Omega^\bullet_{R/\mathbb{Z}_p}$ depends on $\square$

But is indep of the choice of $\square$ in $D_{p\text{-comp}}(\mathbb{Z}_p)$

<u>In fact</u> $(R_0/\mathbb{Z}_p)_{crys}$ site $\to$ obj (E,J) PD-Pair$/\mathbb{Z}_p$ with $R \to E/J$

$\to$ morph: ✓

$\to$ covers $\{(E_i, J_i)\}$ e.g. $R \to \prod E_i/J_i$ f.f. (Zar, no cond)

<u>str sheaf</u> $\mathcal{O}_{crys} : (E,J) \longmapsto E$

<u>Thm</u> (Berthelot, Grothendieck 1960)

$$R\Gamma_{crys}(R_0) \underset{defn}{=} R\Gamma((R_0/\mathbb{Z}_p)_{crys}, \mathcal{O}_{crys}) \underset{qis}{\cong} \Omega^\bullet_{R/\mathbb{Z}}$$

[There is also a canonically defined cx $W\Omega_{R_0}$ with $\cong R\Gamma_{crys}(X_0)$]

§ Thm.

Thm (prel. vers, BS) conj is true.

Idea instead of deforming $\mathbb{F}_p \rightsquigarrow \mathbb{Z}_p$ ($\to$ cryst)
we deform $\mathbb{Z}_p \rightsquigarrow \mathbb{Z}_p[\![q-1]\!]$ ($\to$ q-cryst)

q-crystalline site $A = \mathbb{Z}[\![q-1]\!]$

S sm $\mathbb{Z}_p = A/(q-1)$-alg

$\left(S/A\right)_{q\text{-crys}}$ — obj: q-PD-pair (E, J) over $(A, q-1)$ ($\to$ Georg)
with $S \to E/J$

$\to$ morph: ✓

$\to$ covers: as in crys case.

str sheaf: $\mathcal{O}_{q\text{-crys}}(E, J) = E$

Def: $q\Omega_{S/A} := R\Gamma\left((S/A)_{q\text{-crys}}, \mathcal{O}_{q\text{-crys}} \right)$

Thm (see Fei's talk last time)

$$q\Omega_{S/A} \hat{\otimes}^L_A \mathbb{Z}_p \cong R\Gamma_{crys}(S/\mathbb{Z}_p)$$
$$\underset{\text{Ber}}{\cong} R\Gamma_{crys}(S/pS/\mathbb{Z}_p)$$
$$\underset{\text{Be-Ill}}{\cong} W\Omega^{\bullet}_{S/pS}$$

Setup (for precise vers of Thm)

$$S = \mathbb{Z}_p\text{-alg}, \quad S/pS \text{ f.t. } / \mathbb{F}_p \text{ (say)}$$

Take

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] & P \arrow[r, two heads] & S \\
 & & A \arrow[u] \arrow[r] & \mathbb{Z}_p \arrow[u] \\
 & & q \arrow[r, mapsto] & 1
\end{tikzcd}$$

$(q-1,p)$-completely smooth.

assume $\exists\ \square : A[X_1 \ldots X_n]^\wedge \longrightarrow P$ p-comp étale

$\uparrow$

δ-structure with $\delta(x_i)=0$ — by (1) extends to — δ-str.

$\Updownarrow$

Frob lift

$\rightarrow \quad D_{J,q}(P) = q$-PD envelope of (P,J) (see Georg)

Then $\gamma_i : A[x_1 \ldots x_n]^\wedge \longrightarrow A[x_1 \ldots x_n]^\wedge$

$$x_i \longmapsto q x_i$$
$$x_j \longmapsto x_j \quad (i \neq j)$$

extends to

$$D_{J,q}(P) \xrightarrow{\gamma_i} D_{J,q}(P)$$

$\rightarrow$ can define $q\,\Omega^{*,\square}_{D_{J,q}(P)}$ as above.

Thm (BS, Thm 16.22)

We have
$$q\,\Omega_{S/A} \simeq q\,\Omega^{*,\square}_{D_{J,q}(P)} \quad \text{in } D_{comp}(A)$$

In particular if S is p-completely sm $/\mathbb{Z}_p$

We can take $P = S[\![q-1]\!]$, $J = (q-1)$

Georg $\rightsquigarrow$ it is a q-PD-pair
$$\Rightarrow \quad D_{J,q}(P) = S[\![q-1]\!]$$

and hence
$$q\,\Omega^{*,\square}_{S[\![q-1]\!]/A} \simeq q\,\Omega_{S/A}$$
is independent of the choice of $\square$

<u>pf</u>: the usual (rather BS-dJ, than bro - Ber) $\square$

(cf. Fei's talk)

§ Altogether

- Thm (Fri last time)

$$R\Gamma_{\Delta}\left(S \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] / A \right) \simeq q\Omega_{S/A}$$

where $\mathbb{Z}_p[\zeta_p] = \frac{A}{[p]_q}$

ζ_p p-th prim root of 1

- Thm (étale comparison, BS Thm 9.1 / Rmk 9.3)

$A_{perf} := \mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}$ perfection of A

$\frac{A_{perf}}{[p]_q} \to \mathcal{O} =$ ring of integers in $\mathbb{Q}_p^{cyc} = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$

$q \mapsto \zeta_p$

$X = \operatorname{Spec} S$, $X_{\mathcal{O}} = \operatorname{Spec}(S \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O})$

$$\Rightarrow \quad R\Gamma(X_{\mathcal{O}}, et, \mathbb{Z}_p) \simeq R\Gamma_{\Delta}\left(S \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O} / A_{perf} \right)^{\phi=1}$$

$$= \left(R\Gamma_{\Delta}\left(S \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p] / A \right) \otimes^L_A A_{perf} \right)^{\phi=1}$$

- Assuming $\exists$ framing $\square : A[x_1 \ldots x_n] \to S$ we obtain altogether

$$R\Gamma(X_{\mathcal{O}}, et, \mathbb{Z}_p) = \left(q\Omega^{*,\square}_{S[[q-1]]/A} \otimes_A A_{perf} \right)^{\phi=1}$$

$\to$ integral(!) p-adic version of Grothendieck's Thm from the beginning.